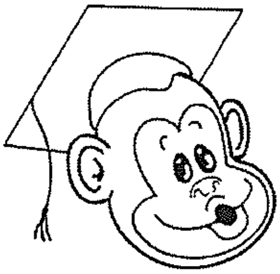


Solution: 20.

One approach is to use algebra to find the answer. If there are *n* seats in the room, then an equation can be solved as follows:

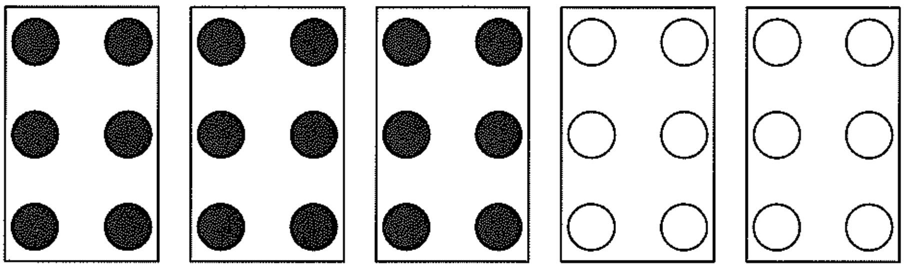
$$\frac{3}{5}n = 18$$

$$\frac{5}{3} \times \frac{3}{5}n = \frac{5}{3} \times 18$$

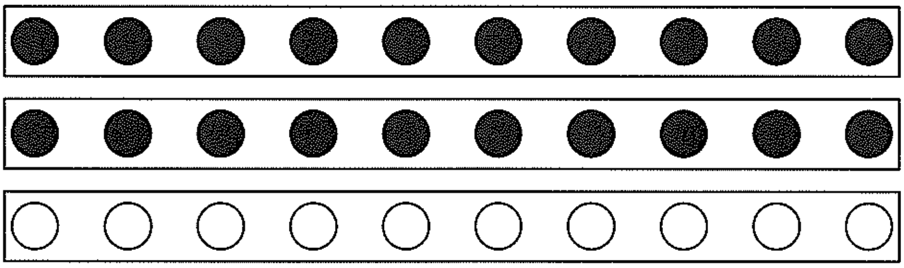
$$n = 30$$

Consequently, if there are 30 seats, then $\frac{2}{3} \times 30 = 20$ students would fill $\frac{2}{3}$ of the seats.

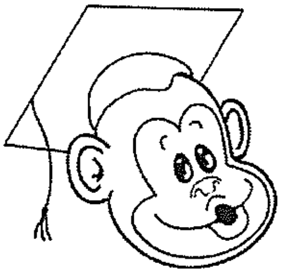
A more intuitive approach might involve a visual representation. If 3/5 of the classroom is 18 seats, then the entire classroom, or 5/5, must be 30 seats, as shown below:



Another visual representation can be used to show that 2/3 of the classroom is 20 seats.



Finally, a purely numeric solution relies on the fact that 3/5 is equivalent to 18/30, which implies that there are 30 seats in the classroom. Then 2/3 × 30 = 20 seats.

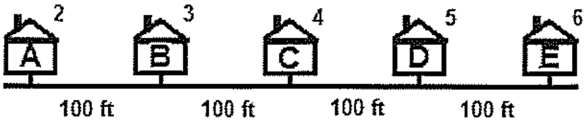


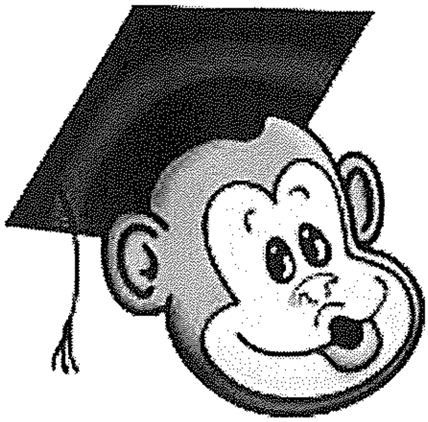
Solution: 12:25.

It's tempting to think that the first time after midnight is 1:21am, or 81 minutes after midnight. Of course, $81 = 9^2$, whereas 121 is 11^2. But there's another possibility: after just $5^2 = 25$ minutes, the clock will read 12:25, and $1{,}225 = 35^2$.

This phenomenon occurs at 12:25 and 1:21, as shown above, but it will also occur at many other times in the future. At $39^2 = 1521$ minutes past midnight, the clock will read 1:21am the following day; at $55^2 = 3025$ minutes past midnight, the clock will read 2:25am two days later; at $81^2 = 6561$ minutes, the clock will read 1:21pm four days later; and so on.

Solution: House D.





If the bus stops at house A, we will have

- 2 children walk 0 ft;
- 3 children walk 100 ft each;
- 4 children walk 200 ft each;
- 5 children walk 300 ft each;
- 6 children walk 400 ft each

Altogether, they walk 5000 ft.

If the bus stops at house B, we will have

- 3 children walk 0 ft;
- 6 children walk 100 ft each;
- 5 children walk 200 ft each;
- 6 children walk 300 ft each

Altogether, they walk 3400 ft.

If the bus stops at house C, we will have

- 4 children walk 0 ft;
- 8 children walk 100 ft each;
- 8 children walk 200 ft each;

Altogether, they walk 2400 ft.

If the bus stops at house D, we will have

- 5 children walk 0 ft;
- 10 children walk 100 ft each;
- 3 children walk 200 ft each;
- 2 children walk 300 ft each;

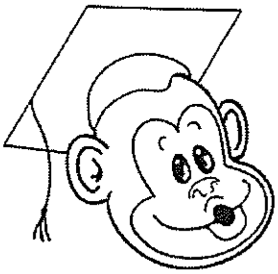
Altogether, they walk 2200 ft.

If the bus stops at house E, we will have

- 6 children walk 0 ft;
- 5 children walk 100 ft each;
- 4 children walk 200 ft each;
- 3 children walk 300 ft each;
- 2 children walk 200ft each.

Altogether, they walk 2600 ft.

Children walk the least if the bus stops at House D.



Solution: 3:36 and 11:20.

The angle between the hour and minute hands equals the product of the hours and minutes at 3:36 and 11.20. In the latter case, the product equals 220, which is the number of degrees between the hands—if you go the long way around!

Let h = hours and m = minutes. The hour hand covers 30° every hour, so at the top of any hour it has traveled $30h$ degrees, and with every passing minute it travels another $m/2$ degrees. Given that the minute hand moves 6° every minute, the number of degrees between the hands is given by either $6m - (30h + m/2)$ or $(30h + m/2) - 6m$. Consequently, the relevant equations are:

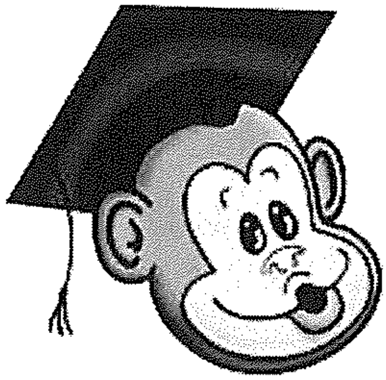
$$hm = 6m - \left(30h + \tfrac{m}{2}\right) \quad \rightarrow \quad m = \frac{60h}{11 - 2h},$$

$$hm = \left(30h + \tfrac{m}{2}\right) - 6m \quad \rightarrow \quad m = \frac{60h}{2h + 11},$$

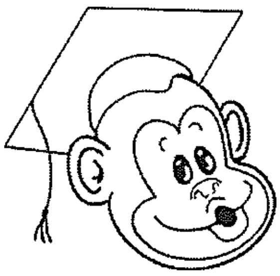
Because m is a positive integer, the first equation implies that $11 - 2h$ must divide evenly into $60h$, and that happens only when $h = 3$. Plugging in $h = 3$ yields $m = 36$, so (3, 36), or 3:36, must have the desired property. Sure enough, there are precisely $3 \times 36 = 108°$ between the hour and minute hands at 3:36.

For the second equation, the right-hand side must be a positive integer, and that happens only when $h = 11$. In that instance $m = 20$, producing (11, 20), or 11:20, as our second time. The fact that you go the "long way around" in this instance arises because the second equation essentially reverses the order of the hour and minute hands.

Solution:



1. 1—*A*
2. 2—*B*
3. 3—*C*
4. 4—*D*
5. 2—*D*
6. 5—*B*
7. 3—*B*
8. 1—*B*
9. 6—*C*
10. 7—*A*
11. 1—*A*
12. 6—*E*
13. 3—*C*
14. 1—*C*
15. 5—*A*
16. 1—*B*
17. 3—*A*
18. 1—*A*
19. 6—*C*
20. 8—*B*
21. 6—*B*
22. 2—*E* (or *C*)
23. 4—*B*
24. 2—*B*.



Solution: 68.256%.

There are three possibilities:

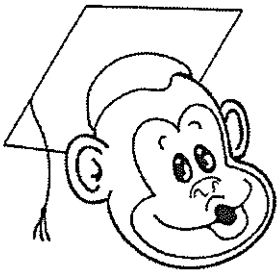
1. Chekmatova wins the first three games, in which case it doesn't matter what happens in the fourth or fifth games.
2. Chekmatova wins two of the first three games and then wins the fourth game, in which case it doesn't matter what happens in the fifth game.
3. Chekmatova wins two of the first four games and then wins the fifth game.

In the first scenario, it doesn't matter what happens in the fourth or fifth games, and the probability of Chekmatova winning the first three games is $(0.6)^3 = 0.216$.

In the second scenario, it doesn't matter what happens in the fifth game. The probability of Chekmatova winning three out of four is $3 \times (0.6)^3 \times (0.4) = 0.2592$. (The 3 at the beginning indicates that her opponent could win any of the first three games.)

In the third scenario, the probability of Chekmatova winning three out of five is $6 \times (0.6)^3 \times (0.4)^2 = .20736$. (The 6 at the beginning of the expression represents the ${}_4C_2$ ways to choose which two games her opponent wins.)

The combined probability is $0.216 + 0.2592 + 0.20736 = 0.68256$, or 68.256%.



Solution: 64 + 64 + 27 + 27 + 27 + 27 + 1 + 1 + 1.

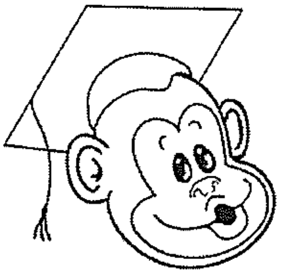
As you may have guessed, the number 239 wasn't chosen at random. It has the unusual property of being the largest number that cannot be represented with *fewer* than nine cubes (23 is the only other number requiring nine).

The only numbers that can be used to add to 239 are 1, 8, 27, 64, 125, and 216 — the cubes of the first six integers, respectively, each of which is less than 239. But the confounding thing about 239 is that the obvious candidates don't work. Starting with 216 or 125 goes nowhere — using these with eight more cubes will get you to 238 but not to 239.

The representation we're looking for starts with 64 and goes like this:

$$239 = 64 + 64 + 27 + 27 + 27 + 27 + 1 + 1 + 1$$

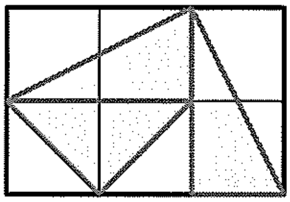
$$= 4^3 + 4^3 + 3^3 + 3^3 + 3^3 + 3^3 + 1^3 + 1^3 + 1^3$$

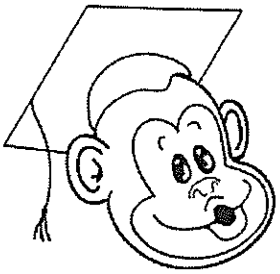


Solution: 27 square centimeters.

The perimeter consists of ten sides of the squares. Since the perimeter is 30 cm, the side length of each square is 30 ÷ 10 = 3 cm.

Further, the gray area can be decomposed into three triangles, as shown below, each of which has an area equal to one of the six squares. (Note that the two pieces of each triangle can be configured to form a square.) Therefore, the gray area is equal to the area of three squares. The area of one square is $3^2 = 9$ sq cm, so the area of three squares is $3 \times 9 = 27$ sq cm.





Solution: 5/9, 5/9, and 5/9.

Because each die contains three different numbers, there are 3 × 3 = 9 possible outcomes when two of these dice are rolled.

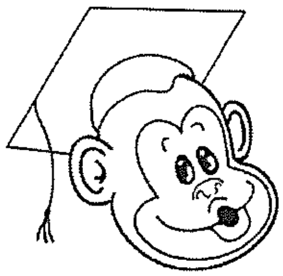
As shown in the tables below, A beats B on 5 out of 9 rolls, 2-1, 4-1, 9-1, 9-6, and 9-8, so the probability that A beats B is 5/9. Similarly, B beats C on 5 out of 9 rolls, 6-3, 8-3, 6-5, 8-5, and 8-7, so the probability that B beats C is also 5/9. Finally, C beats A on 5 out of 9 rolls, 3-2, 5-2, 5-4, 7-2, and 7-4, so the probability that C beats A is, once again, 5/9.

		A		
		2	*4*	*9*
	1	A	A	A
B	*6*	B	B	A
	8	B	B	A

		B		
		1	*6*	*8*
	3	C	B	B
C	*5*	C	B	B
	7	C	C	B

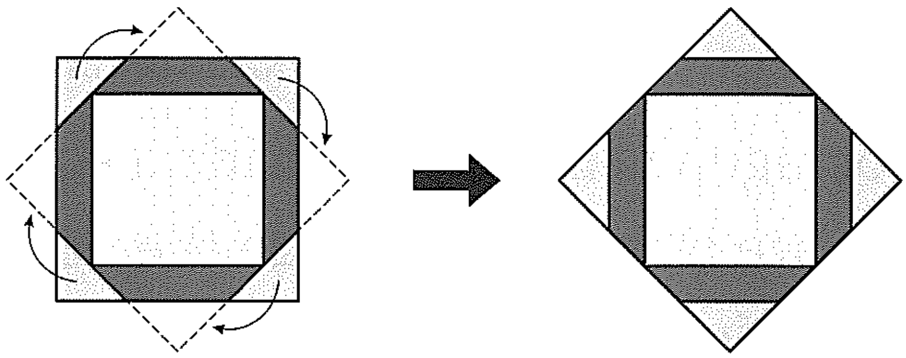
		A		
		2	*4*	*9*
	3	C	A	A
C	*5*	C	C	A
	7	C	C	A

In math, we're used to the inequalities $A > B$ and $B > C$ implying that $A > C$. This relationship is known as the transitive property. But with these dice, that relationship does not hold, so these dice are often referred to as "nontransitive dice." Like the game *Rock, Paper, Scissors*, they don't obey the laws of inequalities that apply to individual numbers.

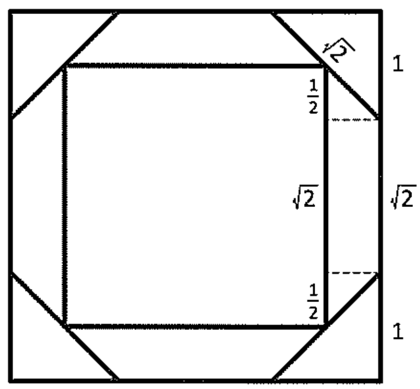


Solution: ½.

One way to see this is to rotate the blue triangles onto the top of the red trapezoids, with the hypotenuse of the triangle flush with the shorter base of the trapezoid. Then it's pretty easy to see that if the four red and blue triangles are folded over, they'd completely cover the yellow square. In other words, the area of the yellow square is equal to the area of red trapezoids and blue triangles combined, so the ratio of the smaller square to the larger square is ½.

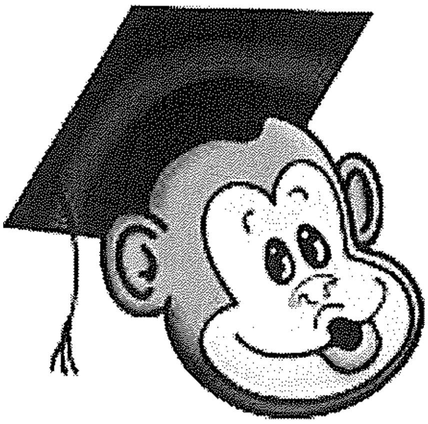


It is also possible to calculate the area of the larger and smaller squares. Start by assuming that the length of the shorter sides of the triangles is 1 unit, as shown below. Then the hypotenuse of each triangle is $\sqrt{2}$, and since the hypotenuse is also a side of the regular octagon, then all sides of the octagon are $\sqrt{2}$.



Consequently, the side length of the larger square is $2+\sqrt{2}$, and the side length of the smaller square is $1+\sqrt{2}$, so their respective areas are $\left(2+\sqrt{2}\right)^2 = 6+4\sqrt{2}$ and $\left(1+\sqrt{2}\right)^2 = 3+2\sqrt{2}$. Dividing the area of the smaller square by that of the larger yields ½.

If those calculations are a little too messy for you, then here is an alternative solution. Note that the length of the diagonal of the smaller square equals the distance between opposite sides of the regular octagon. However, the side length of the larger square is also equal to the distance between opposite sides of the octagon. Therefore, the side length of the larger square is $\sqrt{2}$ times the side length of the smaller square, so the ratio of the areas is ½.



Solution: 5 red beads.

If Julie bought 2 silver beads, she would only have \$8 left to purchase 8 more beads. In that case, she would not be able to buy any blue beads. So, Julie can only purchase 1 silver bead.

That leaves her \$18-\$5=\$13 for 10-1= 9 beads (in red or blue). We will use guess and check to find the number of blue beads and red beads she bought.

Guess #1: 3 red beads and 6 blue beads

Check: the cost is \$3 + \$12 = \$15 > \$13. Since a red bead costs less than a blue bead, we need to increase the guess on the number of red beads.

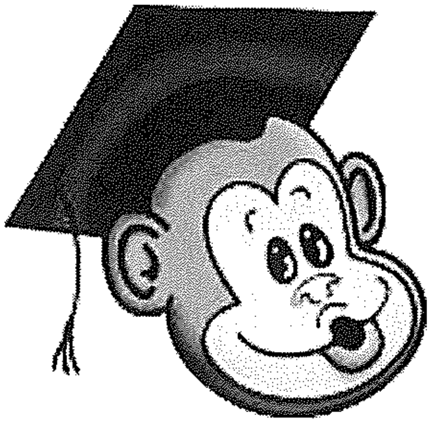
Guess #2: 6 red beads and 3 blue beads

Check: the cost is \$6 + \$6 = \$12 < \$13. Since a red bead costs less than a blue bead, we need to decrease the guess on the number of red beads.

Guess #3: 5 red beads and 4 blue beads

Check: the cost is \$5 + \$8 = \$13. That's it! This is the answer we are looking for.

Therefore, Julie bought 1 silver bead, 4 blue beads, and 5 red beads.

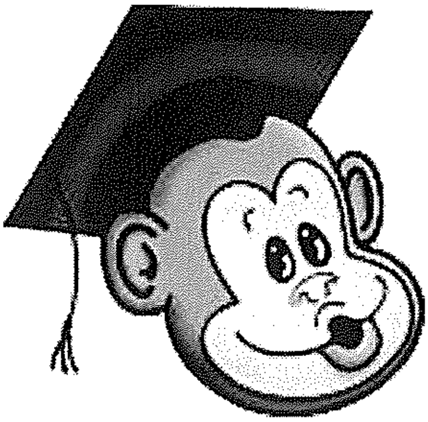


Solution: 14 different ways.

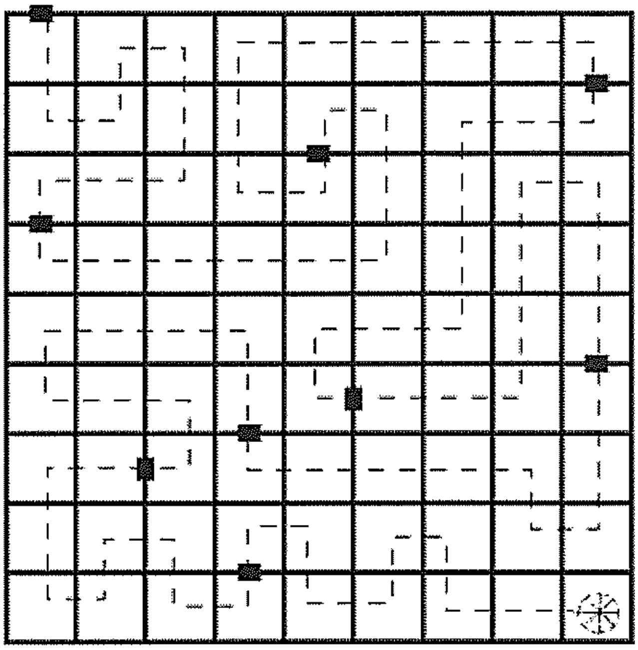
We can count in an organized way. First we put one crayon in the star box and the other 3 in the circle box. This gives us a total of 4 ways. Next we put 2 crayons in each box. This gives us a total of 6 ways. Finally we put 3 crayons in the star box and 1 crayon in circle box. This gives us a total of 4 ways.

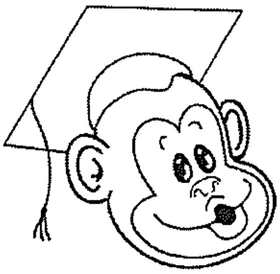
the star box	The circle box
R	Y,B,G
Y	R,B,G
B	R,Y,G
G	R,Y,B
R,Y	G,B
R,G	Y,B
R,B	Y,G
G,B	R,Y
Y,B	R,G
Y,G	R,B
Y,B,G	R
R,B,G	Y
R,Y,G	B
R,Y,B	G

In all, there are 4 + 6 + 4 = 14 different ways to put the crayons in the boxes.



Solution:

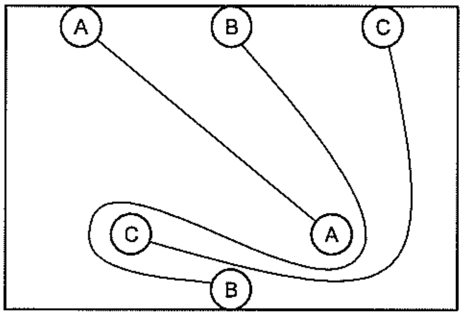


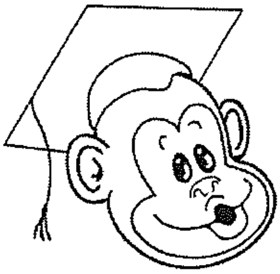


Solution: See figure below.

Many people are baffled by this puzzle, because they want the pieces to connect with short lines. Unfortunately, at least one of the lines has to be reasonably long — it has to wrap around two circles with different letters before finding its match. The figure at the bottom of the page shows one possible solution, but there are others.

Albert Einstein said that anyone who is able to solve this puzzle will do so within two minutes. Further, he believed that anyone who wasn't able to solve it in two minutes would never be able to solve it. That's probably not true. But if you look at this puzzle for several minutes and can't find a solution on your own, put the puzzle aside for a day or two, and then come back to it with a fresh start. Giving your mind a break is often helpful with these kinds of problems.



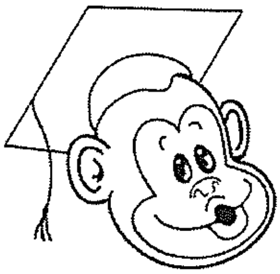


Solution: $5.

If Timmy has $1, then Mara must have $3. But when her mother gives her $20, she will have a total of 3 + 20 = 23, which is more than seven times the amount that Timmy has. (In fact, it is 23 times as much as Timmy has.) A table can be used to investigate other amounts, and the table below shows that the conditions of the problem are satisfied if Timmy has $5.

Number of dollars Timmy has	*Number of dollars Mara has (three times Timmy's amount)*	*Mara's amount after mother gives her $20*	*Ratio of Mara's new amount to Timmy's amount*
1	3	23	23 ÷ 1 = 23
2	6	26	26 ÷ 2 = 13
3	9	29	29 ÷ 3 = 9.67
4	12	32	32 ÷ 4 = 8
5	**15**	**35**	**35 ÷ 5 = 7**
6	18	38	38 ÷ 6 = 6.33

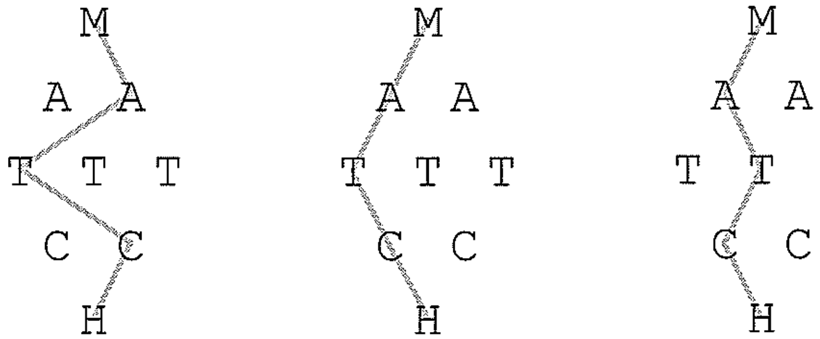
It's also possible to use algebra to solve this problem. Let x = Timmy's amount. Then Mara currently has $3x$ dollars, and she will have $3x + 20$ after her mother gives her $20. Since Mara's new amount is supposed to be seven times Timmy's current amount, this leads to the equation $3x + 20 = 7x$. Solving this equation yields $x = 5$, which verifies the answer in the table.

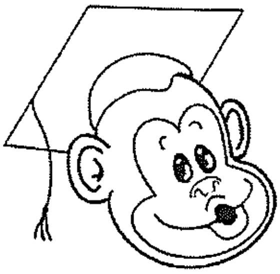


Solution: 12 paths.

To spell the word MATCH, there is one way to choose the M, two ways to choose an A, three ways to choose a T, two ways to choose a C, and one way to choose the H. That results in a total of $1 \times 2 \times 3 \times 2 \times 1 = 12$ possible paths.

Several sample paths are shown below.





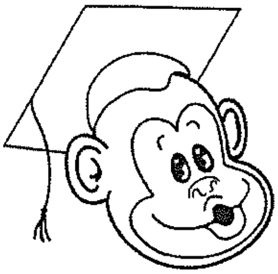
Solution: 10 pins.

A game in which a player alternates strikes and spares is known as a Dutch 200 game. As this name implies, the total score for such a game is 200.

A game in which a player throws no strikes reaches a maximum of 190, if a spare is thrown in each of the first ten frames and a nine is thrown in the first half of each frame and with the bonus ball in the tenth frame. Each frame earns 19 pins, and the total score for the entire game is 190.

The sample score sheet below shows how these total scores result. For the Dutch 200 bowler, notice that the number of pins scored on the first ball in each of the even frames has no effect on the total score. The bowler with no strikes, however, needs to get nine pins on the first ball in every frame to obtain the maximum possible score of 190.

Player	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Spares & Strikes	X 20	8 / 40	X 60	1 / 80	X 100	9 / 120	X 140	4 / 160	X 180	7 / X 200
No Strikes	9 / 19	9 / 38	9 / 57	9 / 76	9 / 95	9 / 114	9 / 133	9 / 152	9 / 171	9 / 9 190



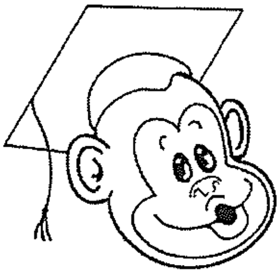
Solution: 10.

The first equation, $x^2 + y^2 = 36$, represents a circle of radius 6 with its center at the origin. The second equation, $xy = 32$, represents a hyperbola that never intersects the circle. Consequently, there are no real solutions to this system of equations. However, there are complex roots, and carrying out the entire calculation using complex numbers would lead to the correct result. Of course, that's messy and difficult, so let's look for a better method.

We have a value for $x^2 + y^2$, we have a value for xy, and we want to know the value of $x + y$. Is there some equation from algebra that relates these three pieces? Indeed, there is—the square of a binomial! That is,

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2 \\ &= x^2 + y^2 + 2xy\end{aligned}$$

Consequently, by substituting the values we know, we get $(x + y)^2 = 36 + 2(32) = 100$. Therefore, $x + y = 10$.



Solution: 12.

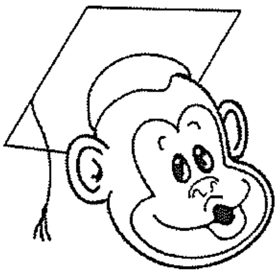
A *partition* is a way of writing an integer as a sum of smaller integers. For instance, 4 + 5 + 6 is a partition of 15, and 11 + 10 + 9 + 7 + 2 is a partition of 39. This problem is concerned with partitions that contain just three addends.

There is only one partition of 3 that contains three addends, namely, 1 + 1 + 1. The trick to this puzzle is realizing that, as the integer increases, the number of partitions either increases or stays the same. For example, there is only one partition of 4 with three addends, 1 + 1 + 2, so the number of partitions stayed the same as the integer increased from 3 to 4. However, there are two partitions of 5 that contain three addends, 1 + 2 + 2 and 1 + 3 + 1, so there are more partitions for 5 than there are for 4.

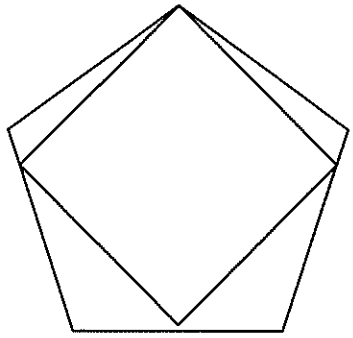
This is helpful, because it means we can limit the number of integers we have to check. Knowing that 3 has only one partition and that 50 has 200, we would suspect that the number we are looking for would be closer to 3 than to 50. So, take an educated guess and try 15. As it turns out, there are 19 partitions of 15 that contain three addends (you can create a list to prove this to yourself). That's too high, since 19 > 15. Searching lower, we might then check 10, which we'd find has just 7 partitions. That's too low, since 7 < 10.

This now significantly limits the range we have to check, and a little more fiddling will lead to the solution. Below are the twelve partitions of 12 with three addends:

1 + 1 + 10	1 + 2 + 9	1 + 3 + 8	1 + 4 + 7
1 + 5 + 6	2 + 2 + 8	2 + 3 + 7	2 + 4 + 6
2 + 5 + 5	3 + 3 + 6	3 + 4 + 5	4 + 4 + 4



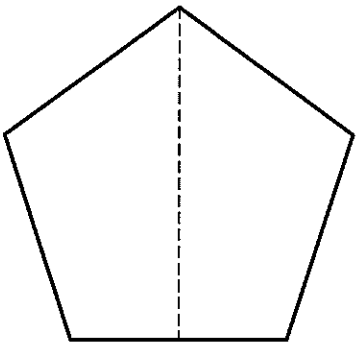
Solution: The square fits inside the pentagon.



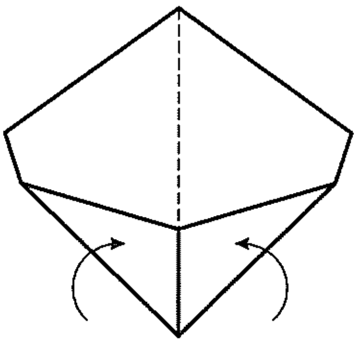
If you are very careful and draw the square perfectly inside the pentagon, you'll see that the fourth vertex of the square lies within the pentagon.

To just see the picture, however, is quite unsatisfying. After all, if your drawing is off by just a little bit, it could appear that the fourth vertex lies on the bottom side or even outside the pentagon. How can you know for sure? You could use trigonometry, but that gets very ugly. Isn't there a more elegant approach?

Perhaps the visual "proof" offered by the following series of folds might convince you.

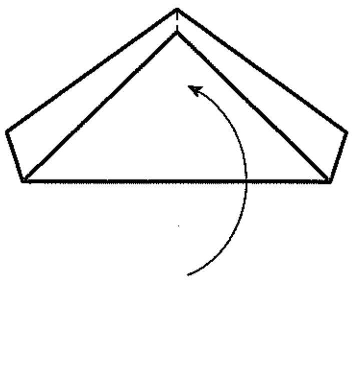


First, fold the pentagon vertically along the midline.



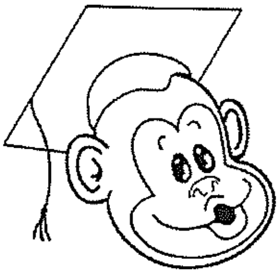
Next, fold the bottom corners of the pentagon so that the base lies on the midline. Doing this will form a right angle at the center of the base.

Why do this? Well, if the fourth vertex of the square lies exactly on the base of the pentagon, then a right angle has to touch the midpoint of the base (which is what was just created). Further, the points at which these two newly formed triangles touch the sides of the pentagon are the points where the vertices of the square must lie.



Finally, fold the bottom portion of the paper so that a horizontal fold connects the points where the triangles from the previous fold meet the sides of the pentagon.

As can be seen, the bottom vertex does not reach the top of the pentagon. This indicates that a square inscribed in the pentagon must have a diagonal that is shorter than the height of the pentagon. Consequently, an inscribed square will lie entirely within the pentagon.

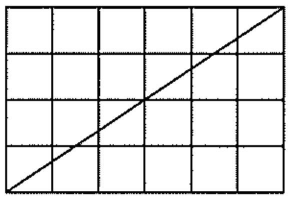


Solution: 5 × 9.

One way to tackle this puzzle is to draw each of the faces, draw the diagonals, and then count the number of tiles. That'll work, but there's more math to be discovered if you look for a pattern.

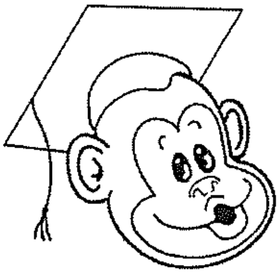
In general, the number of tiles that the diagonal of an $m \times n$ rectangle will cross is given by the expression $m + n - \text{GCD}(m,n)$. It's easy to see why the dimensions need to be added. Each time the diagonal crosses a vertical or horizontal grid line, it touches another tile. However, it's not so easy to see why the greatest common divisor needs to be subtracted. This is best explained with an example.

Look at the 4 × 6 rectangle below. The diagonal passes through a corner where two squares meet. In this case, the diagonal of the 4 × 6 rectangle meets one of these corners twice — once in the middle of the figure and again in the upper-right corner, and GCD(4, 6) = 2. More generally, the number of times this happens is equal to the GCD of the dimensions.



Using this formula, the diagonal of the 5 × 6 face will pass through 5 + 6 – GCD(5, 6) = 10 tiles; the diagonal of the 5 × 9 face will pass through 5 + 9 – GCD(5, 9) = 13 tiles; and, the diagonal of the 6 × 9 face will pass through 6 + 9 – GCD(6, 9) = 12 tiles. Therefore, the 5 × 9 face passes through the most tiles.

This result may seem counterintuitive, since we'd expect the diagonal to pass through the most tiles on the face with the greatest dimensions.

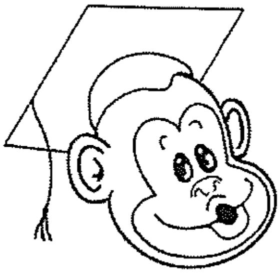


Solution: 120.

Any number that leaves a remainder of 4 when divided into 68 will have two characteristics:

- It will be a factor of 68 – 4 = 64.
- It will be greater than 4. (It's impossible for any number to leave a remainder larger than itself.)

The factors of 64 greater than 4 are 8, 16, 32 and 64, which have a sum of 8 + 16 + 32 + 64 = 120.

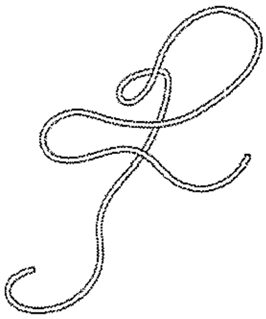


Solution: 0 knots.

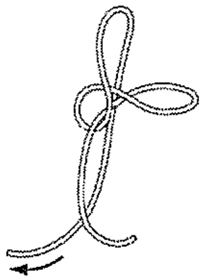
Look at each letter individually. The basic letters (like *r*, *i*, *n*, *e* and the two *s*'s) can be analyzed by inspection, and it can be seen that none of these letters will form a knot when the rope is pulled tight.

There is a twist in each of the two *a*'s. When pulled tight, however, the twists will slip past each other and no knots will be formed.

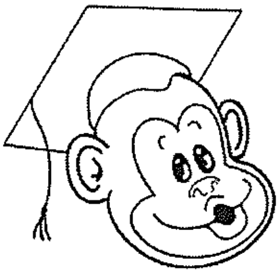
The *B* and *t* appear to be more complex than the other letters. The small loop at the top of the B will obviously not form a knot by itself. When the lower half of the *B* is unwrapped from the vertical stem (as shown below), it can be seen immediately that the rope will straighten out as both ends are pulled. Consequently, no knots will be formed by the *B*.



The *t* consists of five unconnected loops. When the bottom left segment of the *t* is pulled as shown below, the upper left loop will be pulled through the middle loop, straighten and disappear. Once that loop is gone, the lower right segment can be pulled to straighten the upper right loop. The remainder of the rope will then yield no knots when pulled tight.



Consequently, no knots will be formed along any portion of the rope when the ends are pulled.



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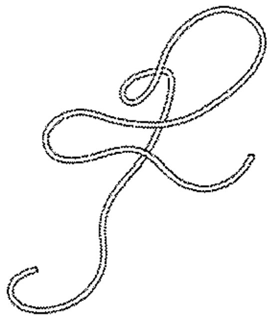
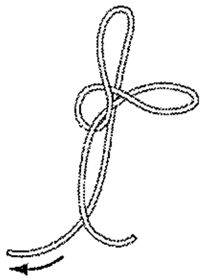
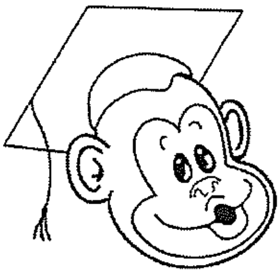


The *t* consists of five unconnected loops. When the bottom left segment of the *t* is pulled as shown below, the upper left loop will be pulled through the middle loop, straighten and disappear. Once that loop is gone, the lower right segment can be pulled to straighten the upper right loop. The remainder of the rope will then yield no knots when pulled tight.



Consequently, no knots will be formed along any portion of the rope when the ends are pulled.



Solution:

Same number four times, $2 \times 2 \times 2 \times 2 = 2^2 + 2^2 + 2^2 + 2^2 = 16$.

Same number three times, $6 \times 2 \times 2 \times 2 = 6^2 + 2^2 + 2^2 + 2^2 = 48$.

Same number two times, $22 \times 6 \times 2 \times 2 = 22^2 + 6^2 + 2^2 + 2^2 = 528$.

Four different numbers, $262 \times 22 \times 6 \times 2 = 262^2 + 22^2 + 6^2 + 2^2 = 69{,}168$.

The solution (2, 2, 2, 2) is rather easy to find. It can be found using a guess-and-check strategy, or you can solve it algebraically with the equation $x^4 = 4x^2$. That equation simplifies to $x^2 = 4$, so $x = 2$. (It turns out that $x = -2$ is also a solution to that equation, but the problem asks for positive integers.)

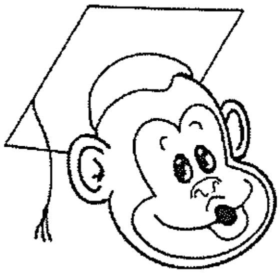
In general, if (a, b, c, d) is a solution to the equation, then $(bcd - a, b, c, d)$ is also a solution. That is, the value of a can be replaced by the value of $bcd - a$. It may not be obvious why this is true, but the following algebra shows that if a is replaced by $bcd - a$, then the equation still holds true:

$$abcd = a^2 + b^2 + c^2 + d^2$$
$$(bcd - a)bcd = (bcd - a)^2 + b^2 + c^2 + d^2$$
$$bcd^2 - abcd = bcd^2 - 2abcd + a^2 + b^2 + c^2 + d^2$$
$$bcd^2 - abcd - (bcd^2 - 2abcd) = bcd^2 - 2abcd - (bcd^2 - 2abcd) + a^2 + b^2 + c^2 + d^2$$
$$abcd = a^2 + b^2 + c^2 + d^2$$

Consequently, if $(a, b, c, d) = (2, 2, 2, 2)$, then $bcd - a = 2 \times 2 \times 2 - 2 = 6$ can replace a to yield the solution (6, 2, 2, 2). This solution uses one number three times.

The order of the numbers is unimportant, so (6, 2, 2, 2) = (2, 6, 2, 2) = (2, 2, 6, 2) = (2, 2, 2, 6). Taking the second of these as $(a, b, c, d) = (2, 6, 2, 2)$, then $bcd - a = 6 \times 2 \times 2 - 2 = 22$ can replace a to yield the solution (22, 6, 2, 2). This solution uses one number twice.

By similar reasoning, $22 \times 6 \times 2 - 2 = 262$ can replace a in (2, 22, 6, 2) to give the solution (262, 22, 6, 2), which uses four different numbers.



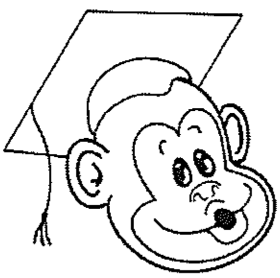
Solution: $9=3+3+3$; $9=3^3\div3$, $9=\sqrt{3^3\times3}$, $9=3^{3!}-(3!)!$.

The easiest solution is the one using only addition, which is shown with the puzzle itself.

The solutions using division and multiplication both require introducing one extra operation—an exponent for the division solution, and two square roots for the multiplication solution.

The solution using only subtraction is the most difficult of the four, because it requires an exponent as well as three factorial symbols, not to mention a fair bit of imagination. The calculations below show that it yields the correct result.

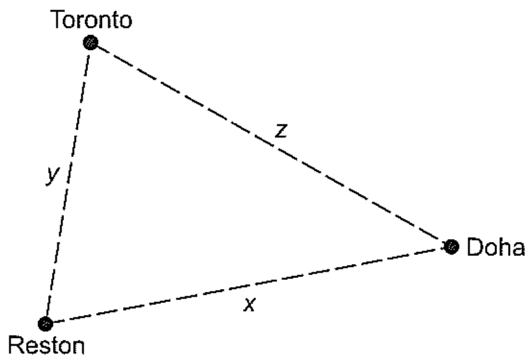
$$\begin{aligned} 3^{3!}-(3!)! &= 3^6-6! \\ &= 729-720 \\ &= 9 \end{aligned}$$



Solution: 13 routes.

The problem can be solved by guess-and-check or with an algebraic approach. The guess-and-check solution is presented first, since it is the method most likely to be used.

In the figure below, assume that there are x direct routes between Reston and Doha, y direct routes between Reston and Toronto, and z routes between Toronto and Doha. But there are also *indirect routes* between the cities. For example, in addition to the x direct routes from Doha to Reston, there are also yz indirect routes, because a passenger could fly from Reston to Toronto, change planes, and then fly from Toronto to Doha. Consequently, the total number of routes from Reston to Doha is $x + yz$, the total number of routes from Reston to Toronto is $y + xz$, and the total number of routes from Doha to Toronto is $z + xy$.



Using a guess-and-check strategy, let $x = 3$, $y = 3$, and $z = 3$. Then there would be $3 + 3 \cdot 3 = 12$ routes between each of the two cities. Increasing x and decreasing y will increase the number of routes from Doha to Reston and decrease the number of routes from Reston to Toronto, so try $x = 4$, $y = 2$, and $z = 3$. This gives $4 + 2 \cdot 3 = 10$ routes between Reston and Toronto (which is one fewer than needed) and $2 + 4 \cdot 3 = 14$ routes between Reston and Doha (which is three fewer than needed). By observation, note that if $x = 5$, the criteria of the problem is satisfied; there will be $5 + 2 \cdot 3 = 11$ routes between Reston and Toronto and $2 + 5 \cdot 3 = 17$ routes between Reston and Doha. With those numbers, there will then be $3 + 2 \cdot 5 = 13$ routes from Toronto to Doha.

Using algebra and some number theory, $x + yz = 17$ and $y + xz = 11$. Adding gives

$$x + yz + y + xz = x + y + z(x + y) = (x + y)(z + 1) = 28,$$

and subtracting gives

$$x + yz - (y + xz) = x - y - z(x - y) = (y - x)(z - 1) = 6.$$

The proper factors of 28 are 1, 2, 4, 7, 14, and the proper factors of 6 are 1, 2, 3. Further, $z + 1$ must be a factor of 28, and $z - 1$ must be a factor of 6. That can only happen if $z = 3$. Plugging that back in gives $x + 3y = 17$ and $y + 3x = 11$, which yields $x = 2$ and $y = 5$. Consequently, the number of routes from Toronto to Doha is $z + xy = 3 + 2 \cdot 5 = 13$.