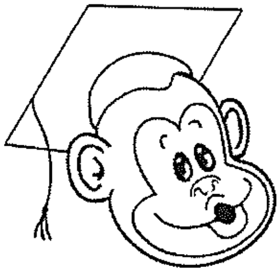


Solution: aced, act, acta, ado, alae, ale, bob, cade, cat, dace, do, el, lea, od.

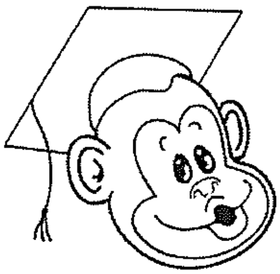
The following are all the ways that 60 can be factored without any factors greater than 26; the corresponding collection of letters is also given. Note that an unlimited number of 1s could be appended to any of these factorizations, so an unlimited number of a's can be added to any of the letter combinations.

Factorization	Letters
$2 \times 2 \times 15$	b, b, o
$2 \times 2 \times 3 \times 5$	b, b, c, e
$2 \times 5 \times 6$	b, e, f
$2 \times 3 \times 10$	b, c, j
3×20	c, t
$3 \times 4 \times 5$	c, d, e
4×15	d, o
5×12	e, l
6×10	f, j

From there, it is just a matter of determining which combinations (plus as many a's as you need) will form legitimate words.

Some of the words in the solution list may not be familiar to you, but all of them are listed in ENABLE (Enhanced North American Benchmark LExicon) and are acceptable words in Scrabble®. Here are definitions for some of the more obscure words in the list.

- acta – document prepared by a notary
- alae – protruding ridge that forms longitudinally on many nematodes (roundworms)
- cade – prickly shrub
- dace – small freshwater fish
- el – elevated railway
- lea – unit of length of thread or yarn
- od – mystic universal force



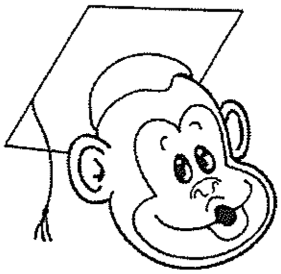
Solution: $9=3+3+3$; $9=3^3\div 3$, $9=\sqrt{3^3\times 3}$, $9=3^{3!}-(3!)!$.

The easiest solution is the one using only addition, which is shown with the puzzle itself.

The solutions using division and multiplication both require introducing one extra operation—an exponent for the division solution, and two square roots for the multiplication solution.

The solution using only subtraction is the most difficult of the four, because it requires an exponent as well as three factorial symbols, not to mention a fair bit of imagination. The calculations below show that it yields the correct result.

$$\begin{aligned} 3^{3!}-(3!)! &= 3^6-6! \\ &= 729-720 \\ &= 9 \end{aligned}$$



Solution: A heart.

The answer for each clue is given below. When all of the answers are colored on the hundreds chart, a heart shape is formed as shown below.

Two positive odd numbers that have a sum of 40 and the largest possible product: **19, 21**.

The smallest square number that is the sum of two non-zero square numbers: **25**.

The next five numbers in the arithmetic sequence 8, 19, 30, **41, 52, 63, 74, 85**.

The maximum possible number of givens in a standard 9 × 9 Sudoku grid that does not render a unique solution: **77**.

Two different odd numbers, one of whose digits are the reverse of the other, whose sum is 154: **59, 95**.

The two prime numbers whose product is 4 less than 5^2: **3, 7**.

In a normal distribution, the percent of values within one standard deviation of the mean: **68**.

The 43rd positive even number: **86**.

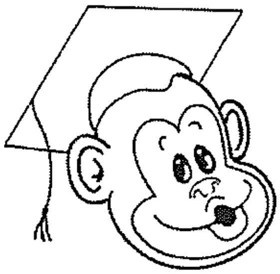
The first four positive multiples of 4: **4, 8, 12, 16**.

The integer lengths of three sides of a right triangle whose area is 600 square units: **30, 40, 50**.

The value of the sum $2^0 + 2^1 + 2^2 + 2^3$: **15**.

The value of the sum $2^0 + 2^1 + 2^2 + 2^3 + 2^4$: **31**.

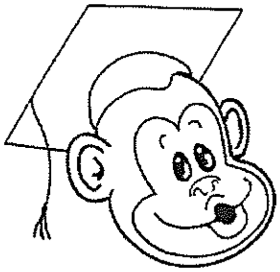
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



Solution: 5 1 9 6 4 2 8 7 3.

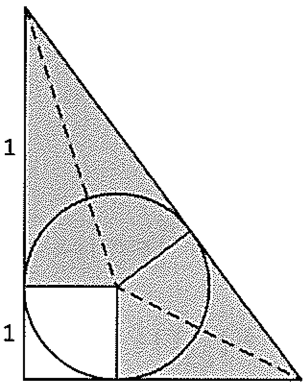
The sequence of triangular numbers is 1, 3, 6, 10, 15, 21, 28, ... For the purpose of this puzzle, though, only 3, 6, 10, and 15 are important. It's not possible to form any other triangular number by adding two single-digit numbers.

One way to get this sequence is to notice that 5 must be either the first or last digit, because there is only one number that can be added to 5 to form a triangular number, namely, 5 + 1 = 6. Once you start with 5 and 1, the third number must be either 2 or 9. If you choose 2, you'll reach a dead end in a few steps. The sequence would continue as 5 1 2 8 7 3, and 3 has the same property as 5: there is only one number, 7, that can be added to it to make a triangular number, and it's already been used. Consequently, the third number must be 9, and then the sequence is uniquely determined: 5 1 9 6 4 2 8 7 3.



Solution: 4, 3, 8/3.

The first thing to note is that the diameter of the circle is 2 units. Since the height of the triangle must be greater than the diameter of the circle, then $a > 2$. (Because we cannot logically distinguish between a and b due to symmetry, then $b > 2$, as well.)



Draw three radii, each perpendicular to a side of the triangle. This divides the triangle into three regions: a blue kite in the upper left, a green kite on the bottom right, and a square of side length 1. The blue kite consists of two triangles, each with height $a - 1$ and base 1, so it has area $a - 1$. Similarly, the green kite consists of two triangles with height $b - 1$ and base 1, so it has area $b - 1$. And the square, obviously, has area 1. Adding these pieces gives the following expression for the area of the triangle: $a + b - 1$.

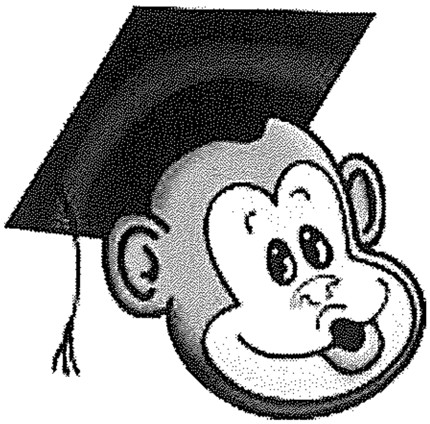
However, we also know that a triangle with height a and base b has an area of $\frac{1}{2}ab$. These two different expressions for the area lead to the following equation:

$$\begin{aligned} \tfrac{1}{2}ab &= a + b - 1 \\ ab &= 2a + 2b - 2 \\ ab - 2a &= 2b - 2 \\ a &= \frac{2b-2}{b-2} \end{aligned}$$

We can now test this equation for integer values of $b > 2$, which lead to the following:

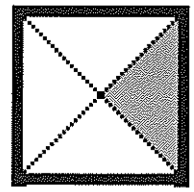
b	a
3	4
4	3
5	$\frac{8}{3}$
6	2

At this point, we can stop. Remember that $a > 2$, so although there are values of $a < 2$ that satisfy the equation, they are not reasonable in the context of the problem. Therefore, we needn't test any values of $b > 6$, and the three possible values of a are 4, 3, and $\frac{8}{3}$.

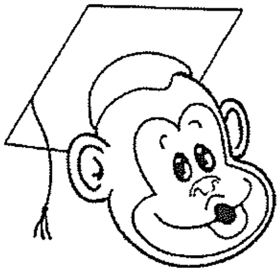


Solution: 4 square inches.

If we only look at one of the square tiles:



The shaded area is equal to ¼ of the total area, which is 4 square inches. So the shaded area is 1 square inch. Together the pattern of tiles contains 4 shaded areas. The total shaded area is then $1 \times 4 =$ 4 square inches.



Solution: Four.

When written out in English, the number four has four letters. Consequently, you'll keep going in an infinite loop.

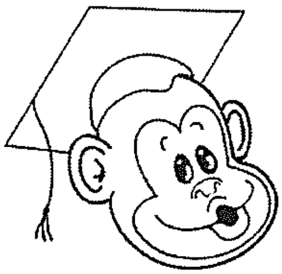
But why do you get to four in the first place? The key to the whole puzzle is that for virtually every number whatsoever, the number of letters is smaller than the number itself. The only exceptions are ONE, TWO, THREE, and FOUR. So no matter which number you start with, you'll eventually get to one of the four numbers in caps above. And what happens if you do? Follow the patterns below:

ONE → THREE → FIVE → FOUR   TWO → THREE → FIVE → FOUR

Note that you'd never get to ONE or TWO unless you chose that number initially.

THREE → FIVE → FOUR   FOUR → FOUR

We're done.



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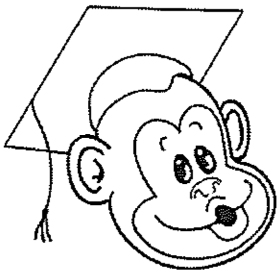


Solution: Susan Saint James works, Mary Beth Hurt does not.

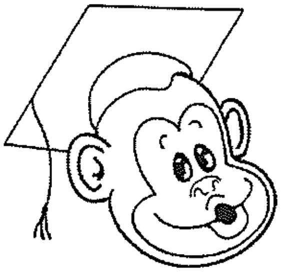
To see why Mary Beth Hurt's name does not work, give the puzzle another look.

In the tens column, there is an R in the first addend, and there is an R in the sum. For that to work, T must be either 0 or 9, depending on whether there was any carrying from the ones column. But in the ones column, notice that there is a T in the sum. This leads to a problem. If T = 0, there were will be carrying from the ones column, and if T = 9, there will be no carrying — exactly opposite of what is needed! Hence, no solution is possible.

```
  MARY
+ BETH
------
  HURT
```

Susan Saint James's name works out a whole lot better. Below is one of several possible solutions:

```
  40461
+ 46513
-------
  86974
```

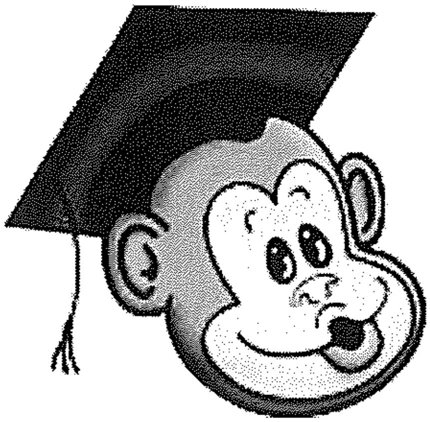


Solution: I missed 6 possibilities.

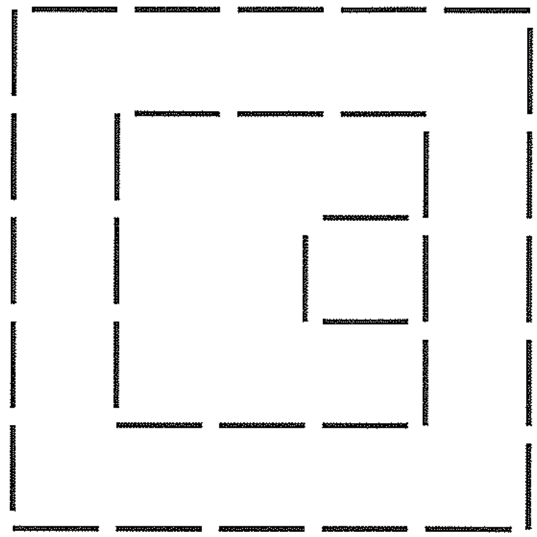
My friend was correct. When four objects are ranked, there are 75 possibilities when ties are included.

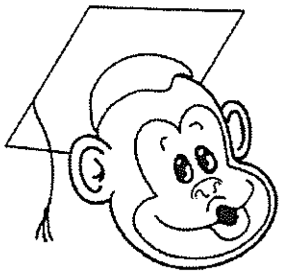
Missing from the list are the six cases in which *two* pairs are tied. Using the notation of the original problem, here they are:

[AB][CD] [AC][BD] [AD][BC] [CD][AB] [BD][AC] [BC][AD]



Solution:

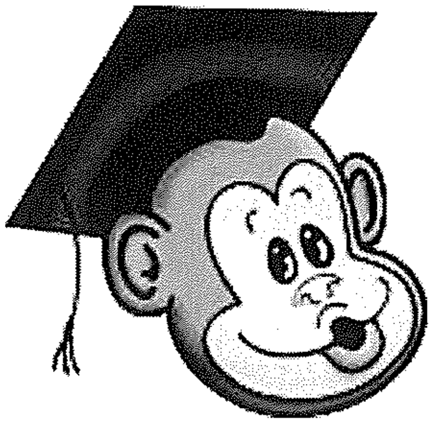




Solution: 48.

You could pick all of the red candies, of which there are 20, and all of the green ones, of which there are 25, and still not have a single brown candy in your hand. That's already 45 candies. You'll then need to pick three brown ones, giving you a total of 48.

This is the number you must pick to be *absolutely certain* that there are three brown candies. However, it's certainly possible — even likely — that you would have three brown candies if you picked fewer. You just can't be absolutely certain that you'll have three brown ones.



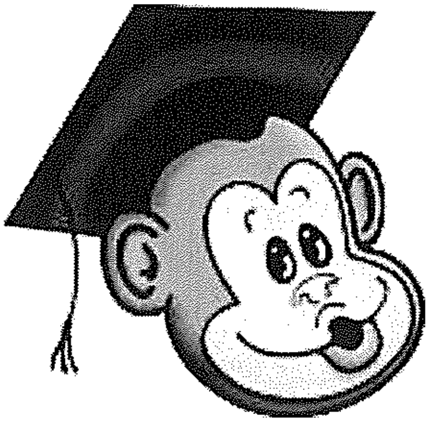
Solution:

The key to the solution is that 2 to the tenth power is 1,024 (that is, over 1,000). With each questions you knock out half the remaining numbers, and after ten questions only the "thought" number is left.

Say the number is 860. The ten questions are:

1. "Is your number greater than 500?" "Yes." Add 250.
2. "Greater than 750?" "Yes." Add 125.
3. "Greater than 875?" "No." Subtract (not 62.5 but the nearest even number) 62.
4. "Greater than 813?" "Yes." Add 31.
5. "Greater than 844?" "Yes."Add (not 15.5 but) 16.
6. "Greater than 860?" "No." Subtract 8.
7. "Greater than 852?" "Yes." Add 4.
8. "Greater than 856?" "Yes." Add 2.
9. "Greater than 858?" "Yes." Add 1.
10. "Greater than 859?" "Yes."

The number is 860.



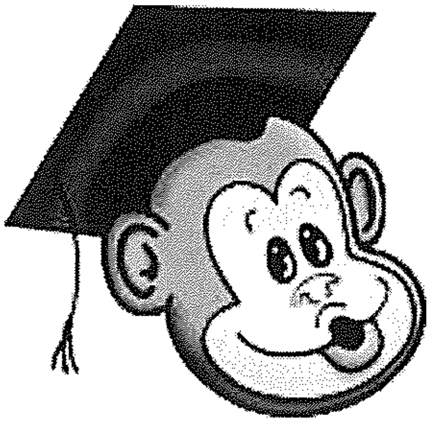
Solution:

The wolf does not eat cabbage, so the crossing can start with the goat.

The man leaves the goat and returns, puts the cabbage in the boat and takes it across. On the other bank, he leaves the cabbage but takes the goat.

He leaves the goat on the first bank and takes the wolf across. He leaves the cabbage with the wolf and rows back alone.

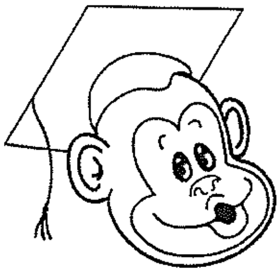
He takes the goat across.



Solution:

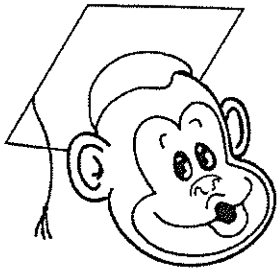
A reasoned: "My friends have white; suppose I have black. Then *B* would say to himself, '*A*'s paper has black and *C* has white. If I have black *C* would see two pieces of black paper and immediately announce he has white. But *C* is silent. Therefore, I announce I have white.' But *B* is silent. Therefore, I announce I have white."

B and *C* reasoned the same way. But *A* could also have reasoned: "In a fair contest, we must all have the same problem to solve. If I see two white pieces of paper, so do they."



Solution: Answers will vary.

Students can check their answers at **http://illuminations.nctm.org/nameletters.**



Solution: The first array, which has 3 rows and 8 columns, cannot be filled to create a magic rectangle.

The integers 1-24 have a sum of 300. This sum can be determined by noticing that the numbers can be regrouped as 12 pairs of numbers with a sum of 25 each:

$$(1 + 24) + (2 + 23) + (3 + 22) + \ldots + (12 + 13) = 300$$

In general, the sum of the first *n* positive integers is given by the following formula:

$$S(n) = \frac{(n)(n+1)}{2} = \frac{n^2+1}{2}$$

Because the sum of the integers 1-24 is 300, to be placed in the 3 × 8 array to form a magic rectangle, the sum of each row would have to be 300 ÷ 3 = 100. This can be done in a number of ways. However, the sum of each column would have to be 300 ÷ 8 = 37.5, which is impossible.

In general, a magic rectangle can be created only if the number of rows and number of columns are both even or both odd; that is, a magic rectangle cannot be created if one is even and the other is odd.

The other arrays can be filled in many different ways. One example of each is shown below.

1	2	3	22	23	24
19	20	21	4	5	6
18	17	16	9	8	7
12	11	10	15	14	13

1	23	3	21	5	19	7	17	16	15	11	12
24	2	22	4	20	6	18	8	9	10	14	13